Solutions to tutorial exercises for stochastic processes

T1. (a) Let A_n be the event that there exists a self-avoiding path starting in 0 of length n of which every vertex has spin 1. Then A_n is measurable, since it only depends on finitely many vertices, i.e., the vertices inside B_n, the ball of radius n around 0. We now have

$$\{0 \longleftrightarrow \infty\} = \bigcap_{n=1}^{\infty} A_n,$$

which is measurable as a countable intersection of measurable sets.

(b) The number of self-avoiding walks of length n starting in 0 can be bounded by 4^n, since in every step of the path there are at most 4 directions to choose from. Let Γ_n be the set of self-avoiding paths of length n starting in 0. It follows that

$$\mathbb{P}_p(0 \longleftrightarrow \infty) \leq \mathbb{P}_p(A_n) \leq \sum_{\gamma\in\Gamma_n} \mathbb{P}_p(\omega_v = 1 \text{ for all } v \in \gamma) \leq 4^n p^n.$$

By taking $p < 1/4$, the above bound goes to 0 for $n \to \infty$. It follows that $\mathbb{P}_p(0 \longleftrightarrow \infty) = 0$ for all $p < 1/4$ and that $p_c \geq 1/4$.

(c) We define a $*$-path to be a sequence of vertices $(x_1, \dots, x_n)$ such that $\|x_i - x_{i-1}\|_\infty \leq 1$ for all $i = 2, \dots, n$. Similarly, we define a $*$-circuit of length n to be a sequence of vertices $(x_0, \dots, x_n)$ such that $x_0 = x_n$ and such that $\|x_i - x_{i-1}\|_\infty \leq 1$ for all $i = 1, \dots, n$. If $0 \not\longleftrightarrow \infty$, then we can find a $*$-circuit π of vertices with spin 0 such that 0 lies in the interior of the circuit. Suppose the circuit has length n, then π contains a vertex $(0, k)$ for some $0 \leq k \leq n$. It follows that there exists a self-avoinding $*$-path starting in $(0, k)$ of length $n-1$ and of which every vertex has spin 0. Let Γ^k_{n-1} denote the set of all such $*$-paths. We have that $|\Gamma^k_{n-1}| \leq 8^{n-1}$, since a star path has 8 directions to choose from in each step. We find

$$\begin{aligned}
1 - \mathbb{P}_p(0 \longleftrightarrow \infty) &= \mathbb{P}_p(0 \not\longleftrightarrow \infty) \\
&\leq \sum_{n=1}^{\infty} \mathbb{P}_p(\text{there exists a } *\text{-circuit } \pi \text{ of vertices with spin 0 around 0}) \\
&\leq \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \mathbb{P}_p(\exists \gamma \in \Gamma^k_{n-1} \text{ such that } \omega_v = 0 \text{ for all } v \in \gamma) \\
&\leq \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \sum_{\gamma\in\Gamma^k_{n-1}} \mathbb{P}_p(\omega_v = 0 \text{ for all } v \in \gamma) \\
&\leq \sum_{n=1}^{\infty} n 8^{n-1}(1-p)^{n-1} \to 0,
\end{aligned}$$

for $p \uparrow 1$. It follows that there exists some $p_0 < 1$ such that $\mathbb{P}_{p_0}(0 \longleftrightarrow \infty) \geq 1/2$, and thus $p_c \leq p_0 < 1$.

T2. (a) Let η and ξ be two configurations with $\eta \leq \xi$. Suppose $\eta(x) = \xi(x) = 0$. Then

$$\begin{aligned} c(x,\eta) &= \exp\left(-\beta\left(\sum_{y\sim x}\mathbb{1}_{\{\eta(x)=\eta(y)\}} - \sum_{y\sim x}\mathbb{1}_{\{\eta(x)\neq\eta(y)\}}\right)\right) \\ &\leq \exp\left(-\beta\left(\sum_{y\sim x}\mathbb{1}_{\{\xi(x)=\xi(y)\}} - \sum_{y\sim x}\mathbb{1}_{\{\xi(x)\neq\xi(y)\}}\right)\right) = c(x,\xi). \end{aligned}$$

Now suppose $\eta(x) = \xi(x) = 1$. Then

$$\begin{aligned} c(x,\eta) &= \exp\left(-\beta\left(\sum_{y\sim x}\mathbb{1}_{\{\eta(x)=\eta(y)\}} - \sum_{y\sim x}\mathbb{1}_{\{\eta(x)\neq\eta(y)\}}\right)\right) \\ &\geq \exp\left(-\beta\left(\sum_{y\sim x}\mathbb{1}_{\{\xi(x)=\xi(y)\}} - \sum_{y\sim x}\mathbb{1}_{\{\xi(x)\neq\xi(y)\}}\right)\right) = c(x,\xi). \end{aligned}$$

(b) We compute

$$\begin{aligned} \varepsilon &= \inf_{\eta} c(x,\eta) + c(x,\eta_x) \\ &= \inf_{\eta} \exp\left(-\beta\left(\sum_{y\sim x}\mathbb{1}_{\{\eta(x)=\eta(y)\}} - \sum_{y\sim x}\mathbb{1}_{\{\eta(x)\neq\eta(y)\}}\right)\right) \\ &\qquad + \exp\left(-\beta\left(\sum_{y\sim x}\mathbb{1}_{\{\eta(x)\neq\eta(y)\}} - \sum_{y\sim x}\mathbb{1}_{\{\eta(x)=\eta(y)\}}\right)\right) \\ &= \min_{0\leq k\leq 2d} \exp\big(-\beta\big(k-(2d-k)\big)\big) + \exp\big(-\beta\big((2d-k)-k\big)\big) \\ &= \min_{0\leq k\leq 2d} 2\cosh\big(\beta(2d-2k)\big) = 2. \end{aligned}$$

Note that $a(x,u) = 0$ whenever $x \neq u$ and $x \nsim u$. Now suppose $x \sim u$. Then

$$\begin{aligned} a(x,u) &= \sup_{\eta} |c(x,\eta) - c(x,\eta_u)| \\ &= \sup_{\eta} \left| \exp\left(-\beta\left(\sum_{\substack{y\sim x\\ y\neq u}}\mathbb{1}_{\{\eta(x)=\eta(y)\}} - \sum_{\substack{y\sim x\\ y\neq u}}\mathbb{1}_{\{\eta(x)\neq\eta(y)\}}\right)\right)(e^{-\beta}-e^{\beta}) \right| \\ &= (e^{\beta(2d-1)})(e^{\beta}-e^{-\beta}). \end{aligned}$$

It follows that

$$M = \sum_{u\neq x} a(x,u) = 2de^{2d\beta}(1-e^{-2\beta}).$$

T3. Consider the bijection $\phi : S \to S$ given by

$$\phi(\eta)(x) = \begin{cases} \eta(x) & \text{if } x \text{ even,} \\ \eta_x(x) & \text{if } x \text{ odd.} \end{cases}$$

Define the generator $\mathcal{L} = \mathcal{L}_\beta$ and its domain $\mathcal{D}(\mathcal{L})$ in the usual way. Then $f \in \mathcal{D}(\mathcal{L})$ if and only if $f \circ \phi \in \mathcal{D}(\mathcal{L})$ since $\phi(\eta_x) = \phi(\eta)_x$ and therefore

$$\sup_\eta |f(\phi(\eta_x)) - f(\phi(\eta))| = \sup_\xi |f(\xi_x) - f(\xi)|,$$

since ϕ is a bijection. Moreover

$$\begin{aligned} c_\beta(x, \phi(\eta)) &= \exp\left(-\beta \sum_{y:y\sim x} (2\phi(\eta)(x) - 1)(2\phi(\eta)(y) - 1)\right) \\ &= \exp\left(\beta \sum_{y:y\sim x} (2\eta(x) - 1)(2\eta(y) - 1)\right) = c_{-\beta}(x, \eta). \end{aligned}$$

Therefore $\mathcal{L}_\beta(f \circ \phi) = \mathcal{L}_{-\beta} f$. We know that the stochastic Ising model with parameter $-\beta > 0$ is ergodic. Therefore there exists a unique invariant measure μ such that for all $f \in \mathcal{D}(\mathcal{L})$, it holds that

$$0 = \int \mathcal{L}_{-\beta} f \mathrm{d}\mu = \int \mathcal{L}_\beta(f \circ \phi) \mathrm{d}\mu = \int \mathcal{L}_\beta f \mathrm{d}\phi_\sharp\mu,$$

where $\phi_\sharp\mu$ is the pushforward measure defined by $\phi_\sharp\mu(A) = \mu(\phi^{-1}(A))$. It follows that $\phi_\sharp\mu$ is an invariant measure for the model with parameter $\beta < 0$. It remains to show that $\phi_\sharp\mu$ is the unique invariant measure. Suppose ν is an invariant measure. Then for all $f \in \mathcal{D}(\mathcal{L})$ we have

$$0 = \int \mathcal{L}_\beta f \mathrm{d}\nu = \int \mathcal{L}_{-\beta}(f \circ \phi) \mathrm{d}\nu = \int \mathcal{L}_{-\beta} f \mathrm{d}\phi_\sharp\nu,$$

so that $\phi_\sharp\nu$ is an invariant measure for the model with parameter $-\beta > 0$. Since this model is ergodic, it follows that $\nu(\phi^{-1}(A)) = \mu(A)$, for all $A \in \mathcal{S}$. Since ϕ is bijective, and since $\phi^{-1} = \phi$, it follows that $\nu(A) = \mu(\phi^{-1}(A))$, for all $A \in \mathcal{S}$. We conclude that $\phi_\sharp\mu$ is the unique invariant measure.